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Derivation of the Boltzmann Equation for a Fermi Gas

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We consider the time evolution of a Fermi gas with two-body interaction. For an initial state ρ which is translation invariant and sufficiently clustering we put $H = H_0 + \lambda V$, we take the limit $\lambda \to 0$, $t \to \infty$ such that $\lambda^2 t = \tau$ and show that (a) the limiting state ρ_τ does not depend on the p-point correlations of ρ for $p > 2$, (b) ρ_τ has vanishing p-point correlations for $p > 2$, and (c) the two-point function that determines ρ_τ satisfies the Boltzmann equation. To avoid nonessential technical difficulties, we consider the case of a Fermi gas on a lattice.

KEY WORDS: Boltzmann equation; approach to equilibrium; $\lambda^2 t$ limit.

1. INTRODUCTION

Hardly any equation in theoretical physics has evoked as much discussion and controversy as the Boltzmann equation. Much of the discussion was and still is centered around the fundamental question at what point and through what assumption the irreversibility in time was introduced. In the original derivation by Boltzmann an essential feature is the Stoszzahlansatz, an assumption about the lack of correlation between the velocities of two colliding particles. This assumption must be made not only at the initial time, but at all times. This clearly very undesirable aspect has triggered many attempts to find more satisfactory derivations.[2,(2)]

A very similar situation exists with respect to the Pauli master equation. In the standard derivation of that equation, an assumption on the

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[2] Of the many papers devoted to the derivation of the Boltzmann equation, we mention in particular the work of two groups, the group of Bogoliubov and coworkers and that of Prigonine and coworkers. See Ref. 1.

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randomness of phases is made, again not just at the initial time but at all times. In 1955 Van Hove[3] published a paper in which a derivation of the master equation was presented without repeated use of a random phase assumption, but with a certain smoothness condition on the initial state. An essential step in this approach is the recognition that in the case of many degrees of freedom diagonal terms in the perturbation expansion are predominant. Another important tool is the so-called $\lambda^2 t$ limit, where the coupling parameter λ tends to zero and the time t tends to infinity, in such a way that $\lambda^2 t$ has a finite limit τ.

One of the difficulties with Van Hove's derivation lies in the fact that some of his basic assumptions are only valid for an infinite system, whereas the formalism only applies to systems of a finite number of degrees of freedom. Even the probabilities appearing in the master equation are not well defined if the system is infinite.

In this paper a derivation is presented of the quantum Boltzmann equation very much along the lines of Van Hove's paper. However, in our case, essential use is made of a formalism, the so-called algebraic approach, that is well suited for the treatment of infinite systems. Furthermore, the distribution functions occurring in the Boltzmann equation are well defined for such systems.

The system considered in this paper is a gas of Fermi particles with two-particle interaction. In order to avoid technical difficulties that have to do with convergence of integrals in momentum space, we deal with a system where the particles are located on an infinite lattice instead of in a continuous space. However, most of the arguments remain valid for the continuous case, as long as one is willing to believe that those integrals converge. The only assumptions made regard the initial state. They are: (1) the initial state is homogeneous, and (2) the initial state satisfies a certain cluster property. The first condition is one of mathematical convenience. It seems feasible though more complicated to treat the nonhomogeneous case along the same lines. The cluster property, i.e., an assumption on the decay of correlation functions, is very essential. It is easy to construct examples of nonclustering initial states that do not behave asymptotically according to a Boltzmann equation.

With these initial conditions we prove that the one-particle distribution function satisfies the Boltzmann equation in the $\lambda^2 t$ limit. The physical interpretation is clear. It means that a very weak interaction acts during a long stretch of time. The variable τ which is the limit of $\lambda^2 t$ is to be interpreted as a rescaled time parameter. It is the time variable occurring in the Boltzmann equation. This method is the mathematical realization of the old idea that, although noninteracting particles do not reach thermal equilibrium, the introduction of an interaction, however slight, will bring the system finally to thermal equilibrium.

We start in Section 2 with some preliminaries on the infinite Fermi gas on a lattice and a discussion of the dynamics, where the two-body interaction is treated as a perturbation of the free dynamics. After a discussion of the assumptions about the initial state ρ, we consider some first- and second-order terms and their $\lambda^2 t$ limit in Section 4. Then in Sections 5 and 6 we treat terms of arbitrary order by means of diagrams and reach two important conclusions. It is shown, firstly, that only the two-point correlation functions of ρ contribute to the $\lambda^2 t$ limit. Secondly, it is found that the state ρ_τ, obtained in the $\lambda^2 t \to \tau$ limit, is quasifree, i.e., has no nonvanishing truncated correlation functions but two-point functions. Finally, in Section 7, it is shown that the one-particle distribution function of this state satisfies, as a function of τ, the quantum Boltzmann equation.

This equation (7.1) differs from the classical Boltzmann equation in two respects. In the first place there is the occurrence of factors $(1 - N(k))$ in the equation, which reflect the exclusion principle. Secondly, because of the weak interaction limit ($\lambda^2 t$ limit) the scattering cross section occurs in Born approximation. The usual equation, with the full scattering cross section would be obtained in the low-density limit, where the density n tends to zero, and t to infinity, in such a way that nt tends to a finite value τ. A derivation would be somewhat more complicated but not essentially different from the derivation presented here.

At the end of the paper we derive some consequences of the quantum Boltzmann equation. We prove the quantum version of the H theorem and show that the Fermi distribution is the only stationary solution of the equation.

2. LATTICE GAS OF FERMI PARTICLES

As mentioned in the Introduction an exact treatment of an infinite system of Fermi particles with two-body interaction gives rise to technical difficulties that are unrelated to the problem of deriving a transport equation. We shall therefore avoid these difficulties and discuss a lattice gas of fermions instead. In this section we shall show that the time evolution of fermions on a lattice with two-particle interaction gives rise to a continuous one-parameter group of automorphisms of the algebra of observables.

At each point x of a three-dimensional lattice we have a creation operator $a(x)^*$ and an annihilation operator $a(x)$. They satisfy the well-known anticommutation relations

$$\{a(x), a(y)^*\} = \delta_{xy}$$
$$\{a(x), a(y)\} = \{a(x)^*, a(y)^*\} = 0$$

Let $\mathfrak{A}$ be the C^* algebra generated by all such creation and annihilation operators.

Translations on the lattice are defined in an obvious manner as a discrete group of automorphisms of $\mathfrak{A}$. To define the dynamics of the system, we must first define the free evolution, i.e., the equivalent of the free evolution of a continuous system.

The one-particle Hilbert space is now $\mathscr{H} = l^2(Z^3)$, i.e., the set of functions f on the lattice, such that

$$\sum_{x \in Z^3} |f(x)|^2 < \infty$$

This allows us to define operators

$$a(f) = \sum_x a(x) f(x)$$

which satisfy the anticommutation relation

$$\{a(f), a(g)\} = \{a(f)^*, a(g)^*\} = 0$$

and

$$\{a(f)^*, a(g)\} = (f, g)$$

For the definition of the free particle evolution it is convenient to consider the Fourier transform:

$$\hat{f}(k) = (2\pi)^{-3/2} \sum_x f(x) e^{-ikx}$$

$\hat{f}$ is periodic in three dimensions with period 2π. We define f_t as follows:

$$\hat{f}_t(k) = \hat{f}(k) e^{i\epsilon_k t}$$

where the kinetic energy ϵ_k is a given even, nonnegative, nonconstant, periodic, and entire function. The free time evolution is now defined by

$$\alpha_t^0(a(f)) = a(f_t)$$

This extends in an obvious way to an automorphism group of $\mathfrak{A}$.

Before introducing an interaction between the particles, we consider some properties of the free evolution:

$$(f, g_t) = \sum_x \overline{f(x)}\, g_t(x) = (2\pi)^{-3} \int_{-\pi}^{\pi} d^3k \overline{\hat{f}(k)} \hat{g}(k) e^{i\epsilon_k t}$$

Since $\overline{\hat{f}(k)} \hat{g}(k) \in L^1$ and ϵ_k is not constant, we conclude that

$$\lim_{|t| \to \infty} (f, g_t) = 0$$

as is the case for the continuous gas. On the other hand, only in the lattice case one can define the wave function δ:

$$\begin{aligned} \delta(x) &= 1, \qquad \text{for} \quad x = 0 \\ &= 0, \qquad \text{for} \quad x \neq 0 \end{aligned}$$

We consider the time evolution of δ:

$$\delta_t(x) = (2\pi)^{-3} \int_{-\pi}^{\pi} d^3k\, e^{i\epsilon_k t} e^{ikx}$$

Since δ_t is the Fourier transform of an infinitely differentiable function, we conclude that $\delta_t \in s(Z^3)$, where $s(Z^3)$ is the set of functions on the lattice that vanish at infinity faster than any power of $|x|$.

A two-particle interaction is defined, for each finite sublattice Λ, by

$$V(\Lambda) = \frac{1}{2} \sum_{x,y \in \Lambda} \phi(x-y) a(x)^* a(y)^* a(y) a(x), \tag{2.1}$$

where the two-body potential ϕ has the following properties:

(a) symmetrical: $\phi(x) = \phi(-x)$

(b) $\sum_x |\phi(x)| < \infty$

These conditions on ϕ are sufficient in order that the interaction gives rise to a strongly continuous group of automorphisms of $\mathfrak{A}$. However, for later purposes we need differentiability of the Fourier transform of ϕ. Then condition (b) must be replaced by

(b′) $\sum_x |x|^p |\phi(x)| < \infty, \qquad$ for some $p > 0$

We shall now multiply $V(\Lambda)$ with a coupling constant λ and consider $\lambda V(\Lambda)$ as a perturbation. The perturbed time evolution is given by the convergent power series in λ

$$\alpha_t^{\lambda,\Lambda}(A) = \sum_{n=0}^{\infty} (i\lambda)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n$$
$$\times \left[\alpha_{t_n}^0(V(\Lambda)), \left[\ldots, \left[\alpha_{t_1}^0(V(\Lambda)), \alpha_t^0(A) \right] \ldots \right]\right] \tag{2.2}$$

We shall prove that $\alpha_t^{\lambda,\Lambda}(A)$ has a limit for $\Lambda \to \infty$. This limit is the time evolution of the system with two-body interaction. The proof is very similar to that of quantum lattice spin systems.[4] (See also Ref. 5.)

We shall assume first that A is a local observable, i.e., A is a polynomial in $a(x)$, $a(y)^*$, for x and y in some finite region. The only complication, as compared to the case of spin systems, is the t dependence in $\alpha_t^0(V(\Lambda))$ on the right-hand side of (2.2). Now

$$\alpha_t^0(V(\Lambda)) = \frac{1}{2} \sum_{x,y \in \Lambda} \phi(x-y) \sum_{z_1 \ldots z_4} a(z_1)^* a(z_2)^* a(z_3) a(z_4)$$
$$\times \overline{\delta_t(z_1 - x)}\; \overline{\delta_t(z_2 - y)}\, \delta_t(z_3 - y) \delta_t(z_4 - x)$$
$$= \sum_{z_1, z_2, z_3, z_4} V_t^{\Lambda}(z_1, z_2, z_3, z_4) a(z_1)^* a(z_2)^* a(z_3) a(z_4)$$

where

$$V_t^\Lambda(z_1,z_2,z_3,z_4) = \frac{1}{2}\sum_{x,y\in\Lambda} \phi(x-y)\,\overline{\delta_t(z_1-x)}\;\overline{\delta_t(z_2-y)}$$
$$\times\,\delta_t(z_3-y)\delta_t(z_4-x)$$

Clearly

$$\lim_{\Lambda\to\infty} V_t^\Lambda(z_1,z_2,z_3,z_4) = V_t(z_1,z_2,z_3,z_4)$$

with

$$V_t(z_1,z_2,z_3,z_4) = \frac{1}{2}\sum_{x,y}\phi(x-y)\,\overline{\delta_t(z_1-x)}\;\overline{\delta_t(z_2-y)}\,\delta_t(z_3-y)\delta_t(z_4-x)$$

As a result of condition (b) there is for fixed t a finite positive number $M(t)$, such that

$$\sup_{|t'|\leqslant t}\sum_{z_1z_2z_3z_4}{}' \left|V_{t'}(z_1,z_2,z_3,z_4)\right| \leqslant M(t) \tag{2.3}$$

where the summation extends over all z_1,z_2,z_3,z_4, with the condition that at least one of them is at the origin. Using (2.3), the repeated commutator in (2.2) can be estimated uniformly in Λ. We get

$$\begin{aligned}
&\left\|\left[\alpha_{t_n}^0(V(\Lambda)),\ldots\left[\alpha_{t_1}^0(V(\Lambda)),\alpha_t^0(A)\right]\ldots\right]\right\| \\
&\quad=\left\|\left[\alpha_{t_n-t}^0(V(\Lambda)),\ldots\left[\alpha_{t_1-t}^0(V(\Lambda)),A\right]\ldots\right]\right\| \\
&\quad\leqslant 2^nN_A(N_A+3)\ldots(N_A+(n-1)3)M(t)^n\|A\|
\end{aligned}$$

where N_A is the number of points of the lattice corresponding to the local observable A. The right-hand side of this inequality is majorized by

$$2^nM(t)^n(N_A+3n)^n\|A\| \leqslant 2^nM(t)^n n!\exp(N_A+3n)\cdot\|A\|$$

so that the nth order term in (2.2) is less in norm than

$$\left(2\lambda tM(t)e^3\right)^n e^{N_A}\|A\| \tag{2.4}$$

where the factor $n!$ is canceled by the time ordering in the integral. We conclude that for A local and given λ, the limit

$$\lim_{\Lambda\to\infty}\alpha_t^{\lambda\Lambda}(A) = \alpha_t^\lambda(A)$$

exists for $t\leqslant t_0$, where t_0 depends on λ, but is independent of N_A. By continuity this result may be extended to all $A\in\mathfrak{A}$. Finally, using the group property, one defines α_t^λ for all t. It is a consequence of the translation invariance of the two-body potential ϕ, that α_t^λ commutes with translations.

3. THE INITIAL STATE

As is customary for infinite systems, a state is a normed positive linear form of the algebra $\mathfrak{A}$. Specifically, if we denote the state by ρ, the expectation value $\rho(A)$ of A has the properties

(a) $\rho(\lambda A + \mu B) = \lambda\rho(A) + \mu\rho(B)$

(b) $\rho(A) \geqslant 0, \quad \text{if} \quad A \geqslant 0$

(c) $\rho(\mathbb{1}) = 1$

A state ρ is invariant under a transformation α if $\rho(\alpha(A)) = \rho(A)$ for all $A \in \mathfrak{A}$. If ρ is invariant under the time evolution α_t^λ then we say that ρ is stationary. If ρ is not stationary, we define a time evolution of ρ by the equation

$$\rho_t^\lambda(A) = \rho\big(\alpha_t^\lambda(A)\big)$$

Clearly ρ_t^λ coincides with ρ at $t = 0$.

We shall be interested in the behavior of ρ_t^λ for large t and weak interaction. We shall study, in particular, the limit $t \to \infty$, $\lambda \to 0$, such that $\tau = \lambda^2 t$ is finite.

We shall have to impose some conditions on the initial state ρ. Some of these conditions are clearly necessary for physical reasons, others are introduced for purely technical reasons or in order to simplify the discussion. The conditions are as follows:

1. Gauge Invariance. This condition says that

$$\rho\big(a(x_1)^* a(x_2)^* \ldots a(x_s)^* a(y_1) \ldots a(y_t)\big) = 0 \qquad \text{unless} \quad s = t.$$

This is a physical condition, based on the fact that only gauge-invariant quantities are observable.

2. Translation Invariance. This restriction is introduced for mathematical convenience.

3. Cluster Properties. As is well known, the expectation value

$$\rho\big(a(x_1)^* \ldots a(x_s)^* a(y_1) \ldots a(y_s)\big)$$

can be expressed in terms of truncated correlation functions:

$$\rho\big(a(x_1)^* \ldots a(x_s)^* a(y_1) \ldots a(y_s)\big)$$
$$= \sum_d (-1)^{P_d} \rho^T(\ldots)\rho^T(\ldots) \ldots \rho^T(\ldots)$$

where the summation extends over all possible divisions d of the set of points $x_1, \ldots, y_s$ into subsets. P_d is the parity of the corresponding permutation. The cluster properties are decay properties of these truncated correlation functions. Because of translation invariance, the truncated functions depend on the differences only:

$$\rho^T(a(x_1)^* \ldots a(x_p)^* a(y_1) \ldots a(y_p)) = f_p(x_2 - x_1, \ldots, y_p - x_1)$$

We define the following:

Cluster Condition I. $f_p \in l^1(Z^{3(2p-1)})$, or

$$\sum_{x_2, \ldots, y_p} |f_p(x_2, \ldots, y_p)| < \infty.$$

As we shall work in momentum space, we consider the Fourier transform $\hat{f}_p$ of f_p. As a consequence of I, $\hat{f}_p$ is a continuous periodic function.

We shall find it convenient to work with creation and annihilation operators for particles of given momentum. We define

$$a(k) = (2\pi)^{-3/2} \sum_x a(x) e^{-ikx}$$

Since the infinite sum on the right-hand side does not exist, $a(k)$ does not exist as a function of k, but has a meaning as distribution. Therefore, also expectation value $\rho(a(k_1)^* \ldots a(k_p)^* a(l_1) \ldots a(l_p))$ must be interpreted as distribution on $[-\pi, \pi]^{6p}$. Let us, on the other hand, consider the truncated "function" $\rho^T(a(k_1)^* \ldots a(l_p))$. As a consequence of translation invariance, there is a factor $\delta^3(k_1 + \cdots + k_p - l_1 - \cdots - l_p)$, so that we can write

$$\rho^T(a(k_1)^* \ldots a(l_p)) = \hat{f}_p(k_2, k_3, \ldots, l_p) \times \delta^3(k_1 + \cdots + k_p - l_1 - \cdots - l_p)$$

Here $\hat{f}_p$ is the Fourier Transform (F.T.) of f_p and hence continuous. We have reached the important conclusion, that the truncated functions in momentum space are continuous, apart from the δ function which expresses momentum conservation.

Of special interest are the two-point functions

$$\rho(a(k)^* a(l)) = N(k) \delta^3(k - l)$$

where $N(k)$ is the one-particle distribution function, and

$$\frac{1}{(2\pi)^3} \int d^3k \, N(k) = n$$

with n the density of the gas.

As will become clear in the following sections continuity of the truncated functions is not enough, but some differentiability is necessary. This is achieved if we require a somewhat stronger cluster property:

Cluster Property II.

$$\sum_{x_1, \ldots, x_{p-1}} |x_1|^\alpha |x_2|^\alpha \ldots |x_{p-1}|^\alpha |f_p| < \infty, \qquad \text{for some} \quad \alpha = 1, 2, \ldots$$

If this is fulfilled $\hat{f}_p(k_1, k_2, \ldots, k_{p-1})$ is simultaneously α times differentiable with respect to each of the variables $k_1, k_2, \ldots, k_{p-1}$.

4. THE $\lambda^2 t$ LIMIT. FIRST- AND SECOND-ORDER TERMS

We shall now study the time evolution of the state ρ_t^λ which equals ρ at $t = 0$. By definition $\rho_t^\lambda(A) = \rho(\alpha_t^\lambda(A))$. Using (2.2) we obtain an expansion in powers of λ,

$$\rho_t^\lambda(A) = \sum_{n=0}^{\infty} (i\lambda)^n \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \times \rho\left(\left[\alpha_{t_n}^0(V), \left[\alpha_{t_{n-1}}^0(V), \left[\ \ldots \left[\alpha_{t_1}^0(V), \alpha_t^0(A)\right] \ldots \right]\right]\right]\right) \tag{4.1}$$

We shall first consider two-point correlations. Let $A = a(f)^* a(g)$; then $\rho_t^\lambda(a(f)^* a(g)) = \int d^3k\, N^\lambda(k,t) \overline{\hat{f}(k)} \hat{g}(k)$. Here $N^\lambda(k,t)$ is the one-particle distribution. We used the fact that the time-evolution α_t^λ commutes with translations, so that ρ_t^λ is translation invariant at all times.

As it is convenient to work in momentum space, we rewrite the interaction as follows:

$$V = \frac{1}{2} \int_{-\pi}^{\pi} d^3k\, d^3l\, d^3m\, d^3n\, v(k-n) \delta^3(k+l-m-n) a(k)^* a(l)^* a(m) a(n)$$

or more symmetrically,

$$V = \frac{1}{4} \int_{klmn} (v(k-n) - v(k-m)) \delta^3(k+l-m-n) \times a(k)^* a(l)^* a(m) a(n) \tag{4.2}$$

where $v(k)$ is the Fourier transform of $\phi(x)$ and $\int_k$ is a shorthand notation for $\int d^3k$. We substitute (4.2) in (4.1), take $A = a(f)^* a(g)$, and consider the term of first order in λ. We obtain

$$\frac{i\lambda}{4} \int_0^t dt_1 \int_{klmn} (v(k-n) - v(k-m)) \delta^3(k+l-m-n)$$
$$\times \int_{k_0 l_0} \rho([a(k)^* a(l)^* a(m) a(n), a(k_0)^* a(l_0)]) \overline{\hat{f}(k_0)} \hat{g}(l_0)$$
$$\times \exp\left[i(\epsilon_k + \epsilon_l - \epsilon_m - \epsilon_n)t_1\right] \exp\left[i(\epsilon_{k_0} - \epsilon_{l_0})t\right]$$

Working out the commutator and using the symmetry in k and l and in m

and n we get the following terms:

$$\frac{1}{2}i\lambda\int_0^t dt_1\int_{klm}\int_{k_0l_0}(v(k-k_0)-v(k-m))\delta^3(k+l-m-k_0)$$
$$\times\rho(a(k)^*a(l)^*a(m)a(l_0))\overline{\hat{f}(k_0)}\,\hat{g}(k_0)\exp\left[i(\epsilon_k+\epsilon_l-\epsilon_m-\epsilon_{k_0})t_1\right]$$
$$+\frac{1}{2}i\lambda\int_0^t dt_1\int_{lmn}\int_{k_0l_0}(v(l_0-n)-v(l_0-m))\delta^3(l_0+l-m-n)$$
$$\times\rho(a(l)^*a(m)a(n)a(k_0)^*)\overline{\hat{f}(k_0)}\,\hat{g}(k_0)\exp\left[i(\epsilon_{l_0}+\epsilon_l-\epsilon_m-\epsilon_n)t_1\right]$$
$$-i\lambda\int_0^t dt_1\int_{lm}(v(l_0-k_0)-v(l_0-m))\delta^3(l_0+l-m-k_0)$$
$$\times\rho(a(l)^*a(m))\overline{\hat{f}(k_0)}\,\hat{g}(k_0)\exp\left[i(\epsilon_{l_0}+\epsilon_l-\epsilon_m-\epsilon_{k_0})t_1\right]\tag{4.3}$$

Notice that, as a consequence of translation invariance the integrands in (4.3) contain a factor $\delta^3(k_0-l_0)$, thereby removing the t dependence in A.

The four-point correlation functions can be expressed in terms of truncated functions:

$$\begin{aligned}\rho(a(k)^*a(l)^*a(m)a(n)) &= \rho^T(a(k)^*a(l)^*a(m)a(n))\\ &\quad+\rho(a(k)^*a(n))\rho(a(l)^*a(m))\\ &\quad-\rho(a(k)^*a(m))\rho(a(l)^*a(n))\end{aligned}$$

Substituting this in (4.3) we obtain

$$\frac{1}{2}i\lambda\int_0^t dt_1\int_{klm}\int_{k_0l_0}(v(k-k_0)-v(k-m))\delta^3(k+l-m-k_0)$$
$$\times\rho^T(a(k)^*a(l)^*a(m)a(l_0))\overline{\hat{f}(k_0)}\,\hat{g}(k_0)\exp\left[i(\epsilon_k+\epsilon_l-\epsilon_m-\epsilon_{k_0})t_1\right]$$
$$+\frac{1}{2}i\lambda\int_0^t dt_1\int_{lmn}\int_{k_0l_0}(v(l_0-n)-v(l_0-m))\delta^3(l_0+l-m-n)$$
$$\times\rho^T(a(l)^*a(m)a(n)a(k_0)^*)\overline{\hat{f}(k_0)}\,\hat{g}(k_0)\exp\left[i(\epsilon_{l_0}+\epsilon_l-\epsilon_m-\epsilon_n)t_1\right]\tag{4.4}$$

where we notice that the contributions from the two-point correlations and the last term in (4.3) cancel each other.

We shall now study the $\lambda^2 t$ limit of the remaining terms, i.e., we let $t\to\infty$ and at the same time $\lambda\to 0$, in such a way that the limit of $\lambda^2 t=\tau$, with τ some positive number. We notice that $\rho^T(a(k)^*a(l)^*a(m)a(n)) = \hat{f}(k,l;m,n)\delta^3(k+l-m-n)$, with $\hat{f}$ continuous. Using this, we rewrite the first term in (4.4) as follows:

$$\frac{1}{2}i\lambda\int_0^t dt_1\,F(t_1)$$

where F is of the form

$$F(t)=\int dk_1\ldots dk_n\,G(k_1\ldots k_n)\exp\left[itE(k_1\ldots k_n)\right]$$

with E analytic in $k_1,k_2,\ldots,k_n$. The continuity and differentiability properties of G depend on the cluster properties of ρ. Assuming only that the truncated functions are in $l^1(Z^{3n-3})$ we can, as in Section 4, conclude that F is continuous and

$$\lim_{|t|\to\infty}F(t)=0$$

As discussed in the Appendix, with somewhat stronger decay properties of the truncated functions, it can be shown that

$$\lim_{|t|\to\infty}t^2F(t)=0$$

This implies in particular that $F\in L^1(\mathbb{R})$ and the limit $t\to\infty$ of the first-order term exists. Consequently, these terms vanish in the $\lambda^2 t$ limit.

We now proceed to second order. One finds

$$\begin{aligned}
&\frac{(i\lambda)^2}{4^2}\int_0^t dt_1\int_0^{t_1}dt_2\int_{k_2l_2m_2n_2}\int_{k_1l_1m_1m_1}\int_{k_0l_0}\left[v(k_2-n_2)-v(k_2-m_2)\right]\\
&\quad\times\delta^3(k_2+l_2-m_2-n_2)\left[v(k_1-n_1)-v(k_1-m_1)\right]\\
&\quad\times\delta^3(k_1+l_1-m_1-n_1)\\
&\quad\times\rho\big(\left[a(k_2)^*a(l_2)^*a(m_2)a(n_2),\left[a(k_1)^*a(l_1)^*a(m_1)a(n_1),\right.\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\left.\left.a(k_0)^*a(l_0)\right]\right]\big)\\
&\quad\times\overline{\hat{f}(k_0)}\,\hat{g}(l_0)\exp\left[i(\epsilon_{k_2}+\epsilon_{l_2}-\epsilon_{m_2}-\epsilon_{n_2})t_2\right]\\
&\quad\times\exp\left[i(\epsilon_{k_1}+\epsilon_{l_1}-\epsilon_{m_1}-\epsilon_{n_1})t_1\right]
\end{aligned}$$

Working out the repeated commutator we get a sum of terms of the typical form,

$$\begin{aligned}
&-\frac{\lambda^2}{16}\int_0^t dt_1\int_0^{t_1}dt_2\int_{k_2l_2m_2}\int_{k_1l_1m_1}\int_{k_0l_0}\left[v(k_2-k_1)-v(k_2-m_2)\right]\\
&\quad\times\delta^3(k_2+l_2-m_2-k_1)\left[v(k_1-k_0)-v(k_1-m_1)\right]\\
&\quad\times\delta^3(k_1+l_1-m_1-k_0)\\
&\quad\times\rho(a(k_2)^*a(l_2)^*a(m_2)a(l_1)^*a(m_1)a(l_0))\overline{\hat{f}(k_0)}\,\hat{g}(k_0)\\
&\quad\times\exp\left[i(\epsilon_{k_2}+\epsilon_{l_2}-\epsilon_{m_2}-\epsilon_{k_1})t_2\right]\exp\left[i(\epsilon_{k_1}+\epsilon_{l_1}-\epsilon_{m_1}-\epsilon_{k_0})t_1\right]
\end{aligned}\tag{4.5}$$

Considering first the contribution of the truncated six-point function and writing

$$\rho^T(a(k_2)^*a(l_2)^*a(m_2)a(l_1)^*a(m_1)a(l_0))$$
$$= \hat{f}(k_2, l_2, m_2, l_1, m_1)\delta^3(k_2 + l_2 + l_1 - m_2 - m_1 - l_0)$$

we obtain

$$-\frac{\lambda^2}{16}\int_0^t dt_1 \int_0^{t_1} dt_2 \int_{k_2 l_2 m_2} \int_{k_1 l_1 m_1} \int_{k_0} \left[v(k_2 - k_1) - v(k_2 - m_2) \right]$$
$$\times \delta^3(k_2 + l_2 - m_2 - k_1)\left[v(k_1 - k_0) - v(k_1 - m_1) \right]$$
$$\times \delta^3(k_1 + l_1 - m_1 - k_0)$$
$$\times \hat{f}(k_2, l_2, m_2, l_1, m_1)\overline{\hat{f}(k_0)}\,\hat{g}(k_0)\exp\left[i(\epsilon_{k_2} + \epsilon_{l_2} - \epsilon_{m_2} - \epsilon_{k_1})t_2 \right]$$
$$\times \exp\left[i(\epsilon_{k_1} + \epsilon_{l_1} - \epsilon_{m_1} - \epsilon_{k_0})t_1 \right] \tag{4.6}$$

This can be written as

$$-\frac{\lambda^2}{16}\int_0^t dt_1 \int_0^{t_1} dt_2\, F(t_1, t_2) \tag{4.7}$$

where F has the form

$$F(t_1, t_2) = \int d^n k\, G(k_1 \ldots k_n)\exp\left[iE_1(k_1 \ldots k_n)t_1 \right]\exp\left[iE_2(k_1 \ldots k_n)t_2 \right]$$

discussed in the Appendix. Under suitable conditions on ρ, one can prove that

$$\lim_{\substack{t_1 \to \infty \\ t_2 \to \infty}} t_1^2 t_2^2 F(t_1, t_2) = 0$$

so that this term vanishes in the $\lambda^2 t$ limit.

Consider next the term obtained by substituting in (4.5)

$$\rho^T(a(k_2)^*a(l_2)^*a(m_1)a(k_0))\rho(a(m_2)a(l_1)^*)$$

One finds

$$-\frac{\lambda^2}{16}\int_0^t dt_1 \int_0^{t_1} dt_2 \int_{k_2 l_2} \int_{k_1 l_1 m_1} \int_{k_0} \left[v(k_2 - k_1) - v(k_2 - l_1) \right]$$
$$\times \delta^3(k_2 + l_2 - l_1 - k_1)\left[v(k_1 - k_0) - v(k_1 - m_1) \right]$$
$$\times \delta^3(k_1 + l_1 - m_1 - k_0)$$
$$\times \hat{f}(k_2, l_2, m_1)N(l_1)\exp\left[i(\epsilon_{k_2} + \epsilon_{l_2} - \epsilon_{l_1} - \epsilon_{k_1})t_2 \right]$$
$$\times \exp\left[i(\epsilon_{k_1} + \epsilon_{l_1} - \epsilon_{m_1} - \epsilon_{k_0})t_1 \right]\overline{\hat{f}(k_0)}\,\hat{g}(k_0) \tag{4.8}$$

which is again of the form (4.7) and vanishes in the $\lambda^2 t$ limit.

Finally we consider a term in (4.5) arising from two-point correlations only. If we substitute $\rho[a(k_2)^* a(l_0)]\rho[a(l_2)^* a(m_1)]\rho[a(m_2)a(l_1)^*]$ in (4.5) we get

$$-\frac{\lambda^2}{16}\int_0^t dt_1 \int_0^{t_1} dt_2 \int_{k_1 l_1 m_1}\int_{k_0}\left[v(k_0-k_1)-v(k_0-l_1)\right]^2 \delta^3(k_1+l_1-m_1-k_0)$$
$$\times N(k_0)N(m_1)\left[1-N(l_1)\right]\overline{\hat{f}(k_0)\hat{g}(k_0)}$$
$$\times \exp\left[i(\epsilon_{k_1}+\epsilon_{l_1}-\epsilon_{m_1}-\epsilon_{k_0})(t_1-t_2)\right]$$
$$=-\frac{\lambda^2}{16}\int_0^t dt_1\int_0^{t_1} dt_2\, f(t_1-t_2) \tag{4.9}$$

As before

$$\lim_{t_1\to\infty}\int_0^{t_1} dt_2\, f(t_1-t_2)=\int_0^\infty du\, f(u)<\infty$$

so that this term is asymptotically proportional to t, and becomes proportional to τ in the $\lambda^2 t$ limit.

We notice that in the surviving term the energy transfer E_2 at t_2 is the opposite of the energy transfer E_1 at t_1. This arises from the fact that the momenta of the created and absorbed particles at t_1 and t_2 are pairwise equal. It is typical for the general case we shall consider in the next section.

5. TERMS OF ARBITRARY ORDER IN λ; DIAGRAMS

If we take in (4.1) $A = a(f_1)^* \ldots a(f_f)^* a(g_1) \ldots a(g_f)$ the nth order term contains the expression

$$\rho\Big(\left[a(k_n)^* a(l_n)^* a(m_n) a(n_n),\right.$$
$$\left[\ \ldots, \left[a(k_1)^* a(l_1)^* a(m_1) a(n_1),\right.\right.$$
$$\left.\left.a(q_1)^* \ldots a(q_f)^* a(s_1) \ldots a(s_f)\right] \ldots \right]\Big) \tag{5.1}$$

In evaluating this we encounter two kinds of contractions.

Type A contractions, arising from the repeated commutator, are contractions between an $a(m_i)$ and $a(k_j)^*$ not both from the same V and not both from A. The rule is the following:

i. The pair $a(m_i), a(k_j)^*$ is replaced by $-(-1)^P\delta^3(m_i-k_j)$, where P is the number of creation and annihilation operators between them as they occur in (5.1).

ii. The total expression is multiplied by $(-1)^n$.

Another kind of contraction, the *type B contraction*, occurs if we write the correlation function in terms of truncated functions. We distinguish:

i. two-point type-B contraction where a pair $a(m_i), a(k_j)^*$ occurring in this order is replaced by $(-1)^P \rho(a(m_i)a(k_j)^*) = (-1)^P(1 - N(m_i))\,\delta^3(m_i - k_j)$, and a pair $a(k_j)^*, a(m_i)$ occurring in this order is replaced by $(-1)^P \rho(a(k_j)^* a(m_i)) = (-1)^P N(m_i)\delta^3(m_i - k_j)$.

ii. $2p$-point contractions ($p = 2, 3, \ldots$) of p creation and p annihilation operators, where these operators are replaced by the corresponding truncated function, multiplied by $(-1)^P$, where P has the obvious meaning.

The various terms may be represented by diagrams. Interactions and the operator A we represent by vertices occurring from right to left in the same order as in (5.1). The vertex representing A has $2f$ directed lines: f ingoing and f outgoing. An interaction vertex has two in-going and two out-going lines. A type-A contraction is a directed dotted line connecting two different vertices. Because of the special form of (5.1) each interaction vertex always has at least one A contraction connecting it with another vertex to its right. A type-B $2p$-point contraction is represented by a circle with p in-going and p out-going lines connecting it with the corresponding vertices.

The diagrams in Fig. 1 represent the two first-order terms in (4.4). The diagrams a, b, and c in Fig. 2 correspond to the second-order terms (4.6), (4.8), and (4.9) calculated in the previous section.

One reaches a considerable simplification if one realizes that between any pair consisting of an absorption operator and a creation operator at *different* vertices there can occur two kinds of contractions, and in calculating the contribution of all possible diagrams these may be added up. This leads to new diagrams, where a directed solid line replaces the sum of an A-type and a B-type contraction, according to the following rules:

	old	value	new	value
A	$\bullet\text{- -}\!\succ\!\text{- -}\bullet\ \ k$	$-(-1)^P$	$\bullet\!\longrightarrow\!\bullet\ \ k$	$-(-1)^P(1 - N(k))$
B	$\bullet\!\rightarrow\!\circ\!-\!\bullet\ \ k$	$(-1)^P N(k)$		
A	$\bullet\text{- -}\!\prec\!\text{- -}\bullet\ \ k$	$-(-1)^P$	$\bullet\!\longleftarrow\!\bullet\ \ k$	$-(-1)^P N(k)$
B	$\bullet\!\leftarrow\!\circ\!-\!\bullet\ \ k$	$(-1)^P(1 - N(k))$		

Each diagram now corresponds to a number of different old diagrams. This

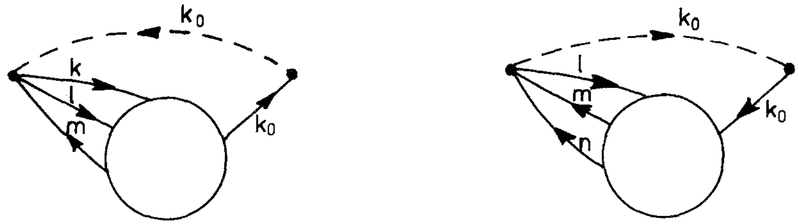

Fig. 1. The diagrams corresponding to the two first-order terms in (4.4).

procedure must, however, be corrected. As mentioned earlier, each interaction vertex must be connected by a type-A contraction with another vertex situated to its right. Therefore, when using the new prescription, for each vertex we have to subtract the term where only B-type two-point contractions link the vertex to its right. As an example we work out the contribu-

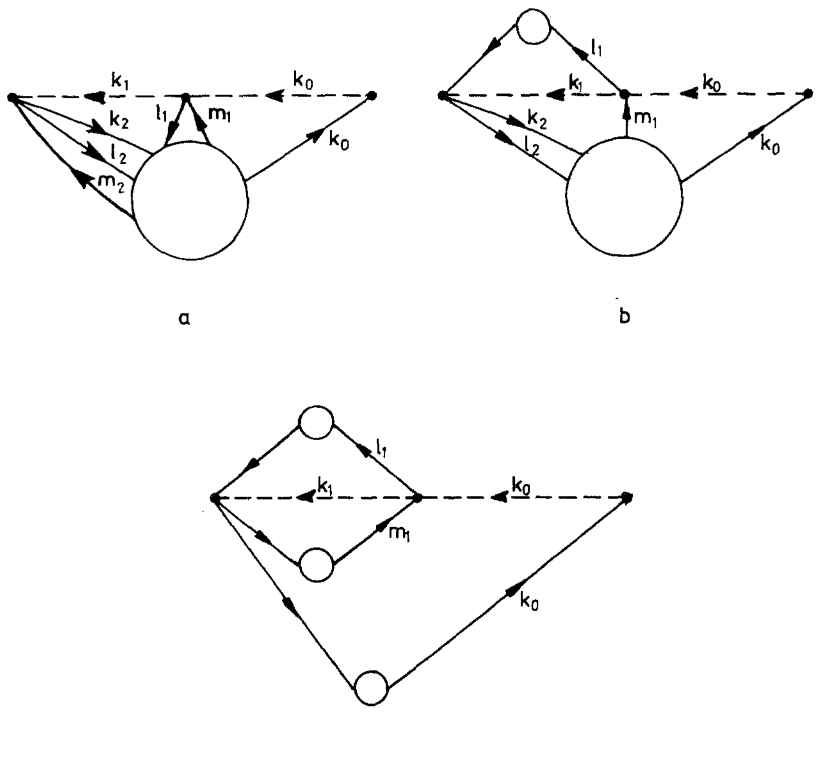

Fig. 2. The diagrams a, b, and c correspond to the second-order terms (4.6), (4.8), and (4.9).

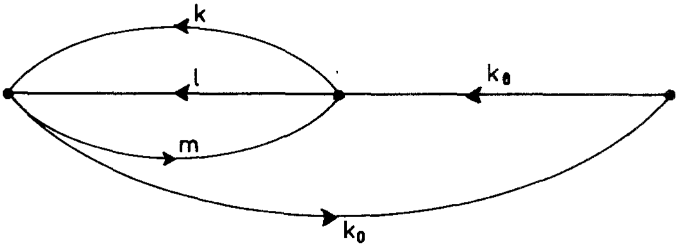

Fig. 3. Example of a diagram with the new internal lines.

tion of the diagram in Fig. 3. We find

$$\begin{aligned}
&\frac{(i\lambda)^2}{16} 8(-1)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_{klm} \left[v(k-k_0) - v(k-m) \right]^2 \delta^3(k+l-m-k_0) \\
&\quad \times \left\{ \left[1 - N(k_0) \right] \left[1 - N(m) \right] N(k) N(l) \right. \\
&\qquad \left. - N(k_0) N(m) \left[1 - N(k) \right] \left[1 - N(l) \right] \right\} \\
&\quad \times \left[-N(k_0) - (1 - N(k_0)) \right] \exp\left[i(t_1 - t_2)(\epsilon_k + \epsilon_l - \epsilon_m - \epsilon_{k_0}) \right]
\end{aligned}$$

The factor 8 is an expression of the fact, that, because of symmetry, a number of terms correspond to the same diagram. This expression represents the sum of several different diagrams of the old kind, among which diagram c of Fig. 2.

These examples suffice to show how an arbitrary nth order term may be calculated.

6. THE $\lambda^2 t$ LIMIT IN nth ORDER

The contribution to $\rho_t^\lambda(A)$, with $A = a(f_1)^* \ldots a(f_f)^* a(g_1) \ldots a(g_f)$, of an nth-order diagram has the general structure

$$(i\lambda)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_n - 1} dt_n F(t_1, t_2, \ldots, t_n, t) \tag{6.1}$$

with

$$F(t_1, \ldots, t_n, t) = \int_{k_1, \ldots, k_p} G(k_1, \ldots, k_p) e^{iE_1 t_1} \ldots e^{iE_n t_n} e^{iEt} \tag{6.2}$$

where the energies $E_1, E_2, \ldots, E$ depend each on some of the momenta $k_1, \ldots, k_p$. It is convenient to rewrite (6.2) in the form

$$F(t_1, \ldots, t_n, t) = \int dE_1 \ldots dE_n dE \hat{F}(E_1, \ldots, E_n, E) e^{iE_1 t_1} \ldots e^{iE_n t_n} e^{iEt} \tag{6.3}$$

As we have seen in the previous examples $\hat{F}$ is a product of some (possibly none) singularities of the form $\delta(E_i + E_j)$ or $\delta(E_i)$ or $\delta(E)$ and a continuous function $\hat{f}$ of the remaining variables. The functions $\hat{f}$ depend on the initial state ρ. As will be discussed in the Appendix ρ must satisfy certain cluster properties in order that the Fourier transform of $\hat{f}$ vanishes sufficiently rapidly at infinity.

We shall first consider the singularities of the kind $\delta(E_i)$. They are due to the diagonal part of the two-particle interaction. An example of a diagram with such a vertex is given in Fig. 4a. As one sees, this diagram is obtained from the diagram of Fig. 4b by adding a vertex at time t_i on an internal line and connecting it with a diagram for a two-point function. If the contribution of that two-point diagram is $N^d(k,t_i)$, the effect of the extra vertex is the same as that of a perturbation term of the form

$$\lambda \int_{kl} N^d(k,t)\left[v(0) - v(k-l)\right]a(l)^* a(l)$$

The effect of the sum of all possible two-point diagrams is then the same of that of a time-dependent external field, corresponding to the following term in the Hamiltonian:

$$\lambda \int_{kl} N(k,t)\left[v(0) - v(k-l)\right]a(l)^* a(l) = \lambda \int_l u(l,t) a(l)^* a(l)$$

Such a term can be incorporated in the unperturbed Hamiltonian. The net

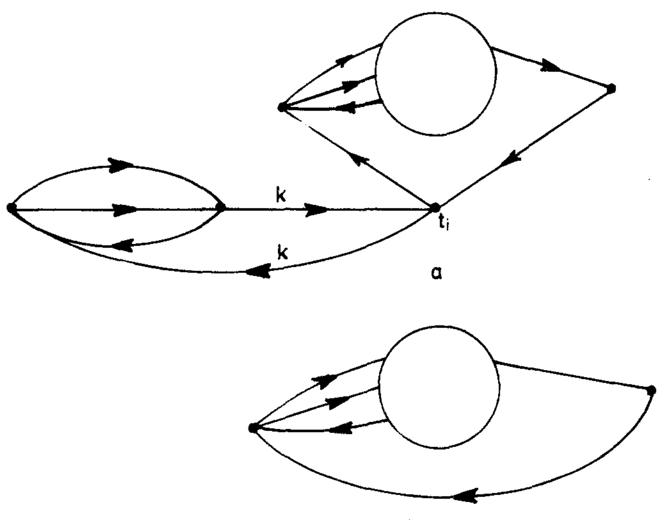

Fig. 4. Diagram a gives an example of a vertex i with a factor $\delta(E_i)$. It is obtained from diagram b by adding a vertex with two-point diagram.

effect is, that everywhere in the energy exponents $\epsilon(k)t$ is replaced by

$$\epsilon(k)t + \lambda \int_0^t u(k,t')\,dt'$$

In the $\lambda^2 t$ limit this extra term vanishes compared to the first and may, therefore, be neglected. We conclude that diagrams that give rise to factors $\delta(E_i)$ may be neglected in $\lambda^2 t$ limit.

The preceding argument does not exclude the possibility that there is a factor $\delta(E)$ corresponding to the A vertex. Let us first assume that there is no such singularity, i.e., that $\hat{F}$ is a nonsingular function of E. The only remaining singularities are then of the form $\delta(E_i + E_j)$. Such a singularity arises when, as a result of two-point contractions the in- and out-going lines of the two vertices i and j have pairwise equal momenta. As a result of the factor $\delta(E_i + E_j)$ in $\hat{F}$ F depends on t_i and t_j only through $t_i - t_j$. If the integration over the variables $t_1, t_2, \ldots, t_n$ were independent from 0 to t, this would give rise to a factor of t. As a result of the time-ordering this is, however, not always the case. Let us consider the fourth-order expression

$$I(t) = \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4\, F(t_1, t_2, t_3, t_4, t)$$

and consider the following cases:

a. $F = f(t_1 - t_2, t_3, t_4, t)$. Keeping $\tau = t_1 - t_2, t_3, t_4$ fixed, the free variable t_2 is integrated from 0 to $t - \tau$, which gives rise to a factor $t - \tau$ in the remaining integrand. Asymptotically for large t this leads to a factor t.

b. $F = f(t_1 - t_2, t_3 - t_4, t)$. As is easily seen, in this case $I(t)$ is asymptotically proportional to t^2.

c. $F = f(t_1, t_2 - t_3, t_4, t)$. Keeping t_1, $t_2 - t_3 = \tau$ and t_4 fixed, the integration of the free variable t_3 extends over the interval $[t_4, t_1 - \tau]$, which gives a factor $(t_1 - t_4 - \tau)$ in the remaining integrand. If f vanishes sufficiently rapidly at infinity we do not obtain a factor t as in case a.

d. $F = f(t_1 - t_3, t_2 - t_4, t)$. As a result of the inequality $t_4 \leqslant t_3 \leqslant t_2 \leqslant t_1 \leqslant t$, we do not get two factors t as in case b, but only one. For a more detailed discussion we refer to the Appendix of this paper.

These examples suffice to draw the following conclusion. The largest power of t in (6.1) occurs for n even, and $\hat{F}(E_1, E_2, \ldots, E_n, E) = \hat{f}(E_1, E_3, \ldots, E_{n-1}, E)\delta(E_1 + E_2) \ldots \delta(E_{n-1} + E_n)$. With (6.3) and (6.1) we obtain

$$(i\lambda)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n\, f(t_1 - t_2, t_3 - t_4, \ldots, t_{n-1} - t_n, t)$$

Its main contribution for large t is

$$(i\lambda)^n \frac{t^{n/2}}{(n/2)!} \int_0^{\infty} d\tau_1 \ldots d\tau_{n/2}\, f(\tau_1, \ldots, \tau_{n/2}, t)$$

In the $\lambda^2 t$ limit, this becomes $\tau^{n/2} h(t)$, where

$$\lim_{t \to \infty} h(t) = 0$$

Consequently even such terms vanish in the $\lambda^2 t$ limit.

A different situation occurs when $\hat{F}$ has a $\delta(E)$ singularity as well. Reasoning as before, the main term for large t occurs for n even, and when $\hat{F} = \hat{f}(E_1, E_3, \ldots, E_{n-1})\delta(E_1 + E_2) \ldots \delta(E_{n-1} + E_n)\delta(E)$. We find

$$(i\lambda)^n \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \, f(t_1 - t_2, t_3 - t_4, \ldots, t_{n-1} - t_n),$$

which for large t becomes

$$(i\lambda)^n \frac{t^{n/2}}{(n/2)!} \int_0^\infty d\tau_1 \ldots d\tau_{n/2} \, f(\tau_1, \ldots, \tau_{n/2})$$

In the $\lambda^2 t$ limit, this term survives and is proportional to $\tau^{n/2}$.

We can now draw two important conclusions.

I. Contributions of diagrams, where for each successive pair of vertices $2i - 1$ and $2i$ there is a factor $\delta(E_{2i-1} + E_{2i})$ contain only two-point contractions. Terms with $2p$-point contractions ($p > 1$) do not contribute in the $\lambda^2 t$ limit. In other words, in the $\lambda^2 t$ limit the truncated correlation functions of ρ with $p > 1$ are irrelevant. If $\rho_{q \cdot f.}$ is the quasifree state which has the same two-point correlations as ρ, then $\rho_t^\lambda(A)$ and $\rho_{q \cdot f \cdot t}^\lambda(A)$ have the same $\lambda^2 t$ limit.

II. Only those diagrams that give rise to a factor $\delta(E)$ can give a nonvanishing contribution in the $\lambda^2 t$ limit. This is automatically fulfilled if $A = a(f)^* a(g)$, owing to momentum conservation. For $A = a(f_1)^* \ldots a(f_f)^* a(g_1) \ldots a(g_f)$, with $f > 1$, a factor $\delta(E)$ can occur only with diagrams of the type shown in Fig. 5, with a contribution which is precisely

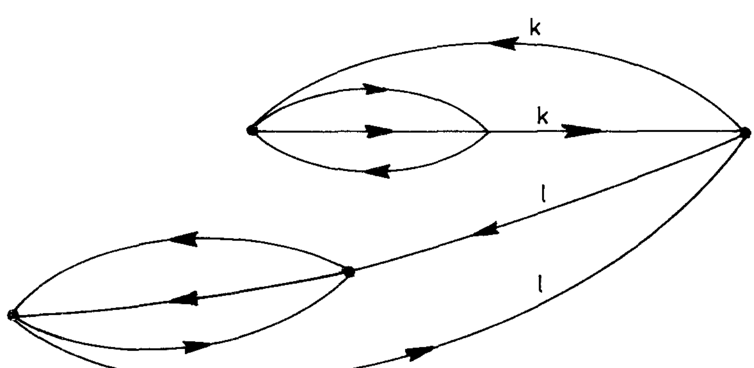

Fig. 5. A diagram that gives rise to a factor $\delta(E)$ at the A vertex.

the product of the contributions of f two-point diagrams. This means that the state ρ_τ, obtained from ρ_t^λ in the $\lambda^2 t$ limit, has only two-point correlations, and hence is *quasifree*.

Our first conclusion expresses the fact that after a sufficiently long time the state forgets the details of its past. The only remaining correlations are those that are consistent with the interparticle forces. According to our second conclusion the only remaining correlations are two-point correlations (one-particle distribution functions).

7. THE TRANSPORT EQUATION

The results of the previous section lead to the conclusion that, without loss of generality, the initial state ρ may be chosen to be quasifree. The $\lambda^2 t = \tau$ limit ρ_τ of ρ_t^λ is again quasifree. Writing

$$\rho_\tau(a(f)^* a(g)) = \int_k N(k,\tau)\overline{\hat{f}(k)}\,\hat{g}(k)$$

we obtain for $N(k,\tau)$ a power series in τ. The coefficient of τ in this series is the derivative of $N(k,\tau)$ with respect to τ at $\tau = 0$. Taking ρ_τ instead of ρ as initial state, we obtain a integro-differential equation involving $N(k,\tau)$.

In order to calculate the term proportional to τ, we proceed as in Section 5 and consider the second-order term. Only such terms contribute in the $\lambda^2 t$ limit, where the energy transfers at t_1 and t_2 are each others opposite. There are only two such diagrams, that of Fig. 3 and the one with all arrows reversed. According to the rules established in Section 5, they give the following contributions:

$$\begin{aligned}
&\frac{1}{2}\lambda^2\int_0^t dt_1\int_0^{t_1} dt_2\int_{klm}|\langle kl|\tilde{V}|mk_0\rangle|^2\delta^3(k+l-m-k_0)\\
&\quad\times\{N(k)N(l)[1-N(k_0)][1-N(m)]\\
&\qquad -N(k_0)N(m)[1-N(k)][1-N(l)]\}\\
&\quad\times\exp[i(t_1-t_2)(\epsilon_k+\epsilon_l-\epsilon_m-\epsilon_{k_0})]
\end{aligned}$$

where $\langle kl|\tilde{V}|mk_0\rangle = v(k-k_0) - v(k-m)$, and a second term, which only differs from this term through the opposite sign in the exponential. Their sum may be written as

$$\frac{1}{2}\lambda^2\int_0^t dt_1\int_0^t dt_2\int dE\,\hat{f}(E)\exp[-i(t_1-t_2)E] = \frac{1}{2}\lambda^2\int_0^t dt_1\int_0^t dt_2\,f(t_1-t_2)$$

In the $\lambda^2 t$ limit we obtain

$$\begin{aligned}\frac{1}{2}\tau\int_{-\infty}^{+\infty}du\, f(u) = \pi\tau\hat{f}(0) = \pi\tau\int_{klm}&|\langle kl|\tilde{V}|mk_0\rangle|^2\delta^3(k+l-m-k_0)\\ &\times\delta(\epsilon_k+\epsilon_l-\epsilon_m-\epsilon_{k_0})\\ &\times\{N(k)N(l)[1-N(m)][1-N(k_0)]\\ &\quad -N(m)N(k_0)[1-N(k)][1-N(l)]\}.\end{aligned}$$

When choosing ρ_τ as initial state, we obtain the transport equation

$$\begin{aligned}&\frac{d}{d\tau}N(k_0,\tau)\\ &\quad=\pi\int_{klm}|\langle kl|\tilde{V}|mk_0\rangle|^2\delta^3(k+l-m-k_0)\delta(\epsilon_k+\epsilon_l-\epsilon_m-\epsilon_{k_0})\\ &\quad\quad\times\{N(k,\tau)N(l,\tau)[1-N(m,\tau)][1-N(k_0,\tau)]\\ &\quad\quad\quad -N(k_0,\tau)N(m,\tau)[1-N(k,\tau)][1-N(l,\tau)]\}\end{aligned}\tag{7.1}$$

As one sees, this is the *quantum Boltzmann equation* where the scattering cross section appears in Born approximation.

It is of interest to draw some immediate consequences of this equation. We define

$$H=\int d^3k\,\{N(k,t)\log N(k,t)+[1-N(k,t)]\log[1-N(k,t)]\}$$

In complete analogy with the derivation of the classical H theorem of Boltzmann one finds

$$\begin{aligned}\frac{dH}{dt}=\frac{\pi}{4}\int_{klmn}&|\langle kl|\tilde{V}|mn\rangle|^2\delta^3(k+l-m-n)\delta(\epsilon_k+\epsilon_l-\epsilon_m-\epsilon_n)\\ &\times\{\log[N(m)N(n)][1-N(k)][1-N(l)]\\ &\quad-\log[N(k)N(l)[1-N(m)][1-N(n)]\}\\ &\times\{N(k)N(l)[1-N(m)][1-N(n)]\\ &\quad-N(m)N(n)[1-N(k)][1-N(l)]\}\end{aligned}\tag{7.2}$$

and hence

$$\frac{dH}{dt}\leqslant 0$$

It also follows from (7.2) that a stationary solution must satisfy the identity

$$N(k)N(l)[1-N(m)][(1-N(n)]=N(m)N(n)[1-N(k)][1-N(l)]$$

for k,l and m,n satisfying momentum and energy conservation. We rewrite this identity as follows:

$$\log \frac{N(k)}{1-N(k)} + \log \frac{N(l)}{1-N(l)} = \log \frac{N(m)}{1-N(m)} + \log \frac{N(n)}{1-N(n)}$$

We conclude that $N(k)$ must have the general form

$$\log \frac{N(k)}{1-N(k)} = a\epsilon(k) + \mathbf{bk} + c$$

or

$$N(k) = \frac{1}{1+e^{-a\epsilon(k)-\mathbf{bk}-c}}$$

For a gas at rest, $\mathbf{b}=0$, $a=-\beta=-1/kT$, and $c=\beta\mu$, with μ the chemical potential.

APPENDIX

In the discussion of the asymptotic t dependence for large t of nth-order perturbation terms it was assumed that the functions $f(\tau_1, \ldots, \tau_s)$ as defined in Section 6 vanish sufficiently rapidly at infinity. In this Appendix we shall give a sufficient condition on f in order that the conclusions of Section 6 are valid, and we shall indicate how this condition derives from decay properties of the truncated correlations of ρ.

As an example, let us consider a term of fourth order,

$$(i\lambda)^4 \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 F(t_1, t_2, t_3, t_4, t)$$

where $F = f(t_1, t_2 - t_3, t_4)$. We assume that f is continuous and

$$s_1^2 s_2^2 s_3^2 |f(s_1, s_2, s_3)| \leqslant M \tag{A1}$$

This implies in particular that f is absolutely integrable. We are interested in the integral

$$I(t) = \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4\, f(t_1, t_2 - t_3, t_4)$$

which can be reduced to the following form:

$$I(t) = \int_0^t dt_1 \int_0^{t_1} d\tau \int_0^{t_1 - \tau} dt_4 (t_1 - \tau - t_4) f(t_1, \tau, t_4)$$

We use (A1) to estimate $|I(t)|$:

$$|I(t)| \leqslant \int_a^t dt_1 \int_a^t d\tau \int_a^t dt_4 (t_1 + \tau + t_4)|f| + C(a)$$

$$\leqslant M \int_a^t dt_1 \int_a^t d\tau \int_a^t dt_4 \frac{t_1 + \tau + t_4}{t_1^2 \tau^2 t_4^2} + C(a)$$

where $a > 0$ and $C(a) > 0$. The integral is easily calculated; we find

$$|I(t)| \leqslant C(a) + 3Ma^{-2} \log \frac{t}{a}$$

Because of the factor λ^4, this term vanishes in the $\lambda^2 t$ limit.

We now consider the general case in nth order. We take a term where $\hat{F}$ in (6.3) contains p factors $\delta(E_i + E_j)$, with $2p \leqslant n$, and a factor $\delta(E)$. Then F in (6.1) depends on $n - 2p$ time variables and p time differences. Introducing these time differences as new integration variables in (6.1), we obtain, after carrying out the integration over the p free variables (i.e., variables not occurring in the integrand), terms of the following general form:

$$I(t) = t^\alpha \int ds_1 \ldots ds_{n-p} (s)^{p-\alpha} f(s_1 s_2 \ldots s_{n-p})$$

with $\alpha = 0, 1, \ldots, p$. Here $(s)^{p-\alpha}$ is a monomial of degree $p - \alpha$ in the variables $s_1, s_2, \ldots, s_{n-p}$. The region of integration is a subset of $[0, t]^{n-p}$.

We shall estimate $|I(t)|$ for large t. The case $\alpha = p = (1/2)n$ can occur only if n is even and for each successive pair of vertices there is a factor $\delta(E_{2i-1} + E_{2i})$. In that case $I(t) = C \cdot t^{(1/2)n}$. For all other cases we estimate $|I(t)|$, assuming as before that f is continuous and

$$s_1^2 s_2^2 \ldots s_{n-p}^2 |f(s_1 \ldots s_{n-p})| \leqslant M \tag{A2}$$

We have

$$|I(t)| \leqslant t^\alpha \int_0^t ds_1 \int_0^t ds_2 \ldots \int_0^t ds_{n-p} (s)^{p-\alpha} |f|$$

$$\leqslant t^\alpha M \int_a^t ds_1 \ldots \int_a^t ds_{n-p} \frac{(s)^{p-\alpha}}{s_1^2 s_2^2 \ldots s_{n-p}^2} + C(a)$$

For $\alpha = p - 1$, the integral is asymptotically $\sim \log t$, for $\alpha \leqslant p - 2$ the integral is at most $\sim t^{p-\alpha-1}$. In all cases, we get for $\alpha < p$

$$\lim_{t \to \infty} t^{-p} |I(t)| = 0$$

This proves that in the nth order all terms vanish in the $\lambda^2 t$ limit except for $\alpha = p = n/2$.

As we have seen, the condition (A2) is crucial for the estimate given above. We shall now show that (A2) holds if the initial state ρ is sufficiently clustering, i.e., if the truncated correlation functions of ρ have sufficient decay properties. The functions f occurring in (A2) are given by integrals of the following form:

$$f(s_1, \ldots, s_m) = \int_{k_1 k_2 \ldots k_n} g(k_1, k_2, \ldots, k_n) e^{iE_1 s_1} \ldots e^{iE_m s_m} \qquad \text{(A3)}$$

where the energies $E_1, E_2, \ldots, E_m$ are each analytic functions of some of the momenta $k_1, k_2, \ldots, k_n$. We shall show that (A2) follows from differentiability properties of g. We show this for the case $M = 1$.

Lemma. Let

$$f(s) = \int_{-\pi}^{\pi} du_1 \ldots du_n \, g(u_1 \ldots u_n) \exp\left[iE(u_1 \ldots u_n) s \right]$$

with g continuous and periodic with period 2π in each of the variables, and E a nonconstant infinitely differentiable periodic function. Then f is continuous and vanishing at infinity.

If an addition g is twice continuously differentiable, then $s^2 f(s)$ vanishes at infinity, so that *a fortiori*

$$s^2 f(s) \leqslant M$$

For the proof see, e.g., Ref. 6. The function $g(k_1 \ldots k_n)$ in (A3) is a product of some truncated correlation functions and some factors $v(k_i - k_j)$. Concerning $v(k)$ we assume it to be sufficiently differentiable; in the special case that the interaction is of finite range, it is even entire. The differentiability properties of g depend, therefore, on the truncated functions. As discussed in Section 4 these follow from suitable decay properties of the truncated correlation functions in x space.

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